NACA TN 2344

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2344

METHOD FOR CALCULATING DOWNWASH FIELD DUE TO LIFTING SURFACES AT SUBSONIC AND SUPERSONIC SPEEDS

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Washington
April 1951

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METHOD FOR CALCULATING DOWNWASH FIELD DUE TO LIFTING SURFACES AT SUBSONIC AND SUPERSONIC SPEEDS

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SUMMARY

A method utilizing source singularities is presented for obtaining the linearized downwash field due to lifting wings of infinitesimal thickness at subsonic and supersonic speeds. The distribution function for the source singularities is specified by the loading on the wing. The method is applied to derive generalized formulas for the downwash field due to uniformly loaded swept and rectangular wings at subsonic and supersonic speeds. The utilization of these formulas to obtain the downwash due to wings of arbitrary loading is indicated. An example of the procedure is given in which specific formulas are derived for the downwash field due to a rectangular wing at supersonic speeds for a uniform loading and for a linear chordwise variation in loading.

INTRODUCTION

Several methods based on linearized theory are available to obtain the downwash field due to lifting surfaces at subsonic and supersonic speeds for use, as an example, in stability calculations. The calculation of the downwash field at subsonic speeds has relied almost exclusively on Prandtl's lifting-line theory, which is based on the concept of a horseshoe vortex (for example, reference 1). Present methods for calculating the downwash field at supersonic speeds are those utilizing conical flows (reference 2), potential doublets (reference 3), vortices (references 4 and 5), and pressure doublets (references 6 and 7). The integrations required of the foregoing vortex or doublet singularities or in the conical-flow method in order to obtain exact solutions of the linearized equations for lifting surfaces have generally been found to be difficult; therefore practice has usually had recourse to approximate methods based on lifting-line theories (references 5 and 7).

The present report, prepared at the NACA Lewis laboratory, indicates a method that is intended to facilitate the computations for obtaining the exact linearized downwash field due to lifting surfaces

at subsonic and supersonic speeds. The method utilizes source singularities with the distribution function specified by the loading on the surface. The method is applied herein to derive formulas for the downwash field due to uniformly loaded swept and rectangular wings of infinitesimal thickness at subsonic and supersonic speeds. The utilization of these formulas to obtain the downwash field due to wings of arbitrary loading, by means of the correspondence relations presented in reference 8, is then indicated. An example of the procedure is given in which specific formulas are derived for the rectangular wing at supersonic speeds for a uniform loading and for a linear chordwise variation in loading.

SYMBOLS

The following symbols are used in this report:

A,B,C_1,C_2 $D_1,D_2,\ldots D_5$		refer to regions bounded by foremost Mach aft cone corners of rectangular wing at supersonic speed (plan view of regions in fig. 3)
a		constant used to describe prescribed linear chordwise variation in loading
B	=	$\sqrt{M^2 - 1}$ (also used to refer to Mach cone region as indicated in fig. 3)
B_1	=	$\sqrt{1 - M^2}$
b		arbitrary constant
c		chord for rectangular wing
h		wing semispan
I_1		refers to integral and solution expressed by equations (25b) and (25e), respectively
I_2		refers to integral and solution expressed by equations (25c) and (25g), respectively
K		constant in equation (1) ($K = \frac{1}{2\pi}$ at subsonic speeds; $K = \frac{1}{\pi}$ at supersonic speeds)

M		free-stream Mach number
m		$\cot\Lambda$, where Λ is angle of sweep of wing leading edge (fig. 1)
r	=	$\sqrt{(x-\xi)^2-B^2\left[(y-\eta)^2+z^2\right]}$
r'	=	$\sqrt{(x'-\xi')^2-B^2(y'^2+z'^2)}$
u,v,w		disturbance velocities of fluid in flow field along x-, y-, and z-axes, respectively (fig. 1)
u_0		value of u on upper surface of wing for uniform prescribed loading
$\bar{w}$		refers to downwash due to left half of wing
w_0		refers to downwash due to uniform prescribed loading on wing
w_1		refers to downwash due to linear chordwise variation in prescribed loading on wing
w_L		contribution to downwash of continuous portion of leading edge for semi-infinite oblique wing with uniform loading shown in figure 1; also represents downwash due to semi-infinite line source originating at leading edge of center section
w_{LD}		contribution to downwash of discontinuity of leading edge at origin in semi-infinite oblique wing with uniform loading shown in figure 1
$w_{n,k}$		downwash due to term in series formed by expressing u on wing as function of x, y, and u_0
w_S		contribution to downwash of streamwise side edge for semi-infinite oblique wing with uniform loading shown in figure 1; also represents downwash due to semi-infinite streamwise line source originating at leading edge of center section
w_{x_0,y_0}; $\bar{w}_{x_0,y_0}$		represents downwash due to semi-infinite line source originating at (x_0,y_0) on right and left halves of wing, respectively

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x,y,z		rectangular coordinates with origin at leading edge of center section (fig. 1)
x',y',z'		oblique coordinates related to rectangular coordinates according to equations (5)
x_0,y_0		rectangular coordinates indicating origin of semi-infinite line source
y_a	=	$y-h$
y_b	=	$y+h$
$(\Delta w_0)_T$		contribution to downwash of wing cut-off at trailing edge for uniform loading
ϵ		infinitesimal distance in y-direction across side edge
$\xi,\eta,\zeta,\xi',\eta',\zeta'$		auxiliary variables used to replace x, y, z, x', y', and z', respectively
ξ_a	=	$B\sqrt{{y_a}^2+z^2}$
ξ_b	=	$B\sqrt{{y_b}^2+z^2}$
ξ_d	=	$\sqrt{\lvert B^2\rvert (y^2+z^2)}$
ξ'_d	=	$\sqrt{\lvert B^2\rvert (y'^2+z'^2)}$
ξ_2,ξ'_2		upper limit for integral in equation (7) (See discussion following equation (7).)
τ		region of integration (See discussion following equations (1) and (3).)
Ω		function representing solution of linearized partial differential equation (2)

Subscripts:

A,B,C_1,C_2 $D_1,D_2,\ldots D_5$	refer to corresponding regions indicated in figure 3

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c_r	refers to chord at center section of swept wing
c_t	refers to chord at tip section of swept wing
L,S,T,LD,TD	refer to continuous portion of leading edge, side edge, trailing edge, discontinuity in leading edge, and discontinuity in trailing edge, respectively
u,l	refer to upper and lower surfaces of airfoil, respectively

Single or successive subscript coordinates indicate partial differentiation with respect to subscript variable.

x^s, y^t	indicates partial differentiation with respect to x and y, s and t times, respectively

BASIC THEORY

The analysis is based on the usual assumptions for thin airfoils in the linearized potential field. A solution for the disturbance parameters can thus be obtained by integrations of source and doublet singularities in the plane of the wing ($z = 0$). The basic equation is (for example, references 9 and 10)

$$\Omega(x,y,z) = -\frac{K}{2}\iint_{\tau}\left[\frac{1}{r}\left(\frac{\partial\Omega_u}{\partial z} - \frac{\partial\Omega_l}{\partial z}\right) + (\Omega_u - \Omega_l)\,\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right] d\xi\, d\eta \tag{1}$$

where the function Ω is a solution of the linearized partial differential equation for subsonic and supersonic flows

$$-B^2\Omega_{xx} + \Omega_{yy} + \Omega_{zz} = 0 \tag{2}$$

In equation (1), $r = \sqrt{(x-\xi)^2 - B^2\left[(y-\eta)^2 + z^2\right]}$ and the region τ includes the entire $z = 0$ plane that can influence the point (x,y,z). At subsonic speeds, the factor K is equal to $1/2\pi$. At supersonic speeds, the factor K is equal to $1/\pi$ and only the finite part of the integral is used. It is important to note that the function Ω can represent either the velocity potential or any of its derivatives, and

if all these values vanish sufficiently far ahead of the wing, the integrals of Ω are solutions of equation (2). In case the derivative of Ω becomes infinite at one or more points, the substitution of this derivative for Ω in equation (2) depends on the condition that the isolation of each singularity yields a finite integrand in the limit.

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SOURCE DISTRIBUTION FOR DOWNWASH FIELD

A lifting wing of infinitesimal thickness is considered. The perturbation velocity u vanishes everywhere in the $z = 0$ plane except on the wing itself, and the perturbation velocity v vanishes everywhere in the $z = 0$ plane except on the wing and in the wake. In equation (1), letting $\Omega = w$ and noting that

$$w_u - w_l = 0$$

$$w_z = B^2 u_\xi - v_\eta$$

$$(u_\xi)_u = -(u_\xi)_l$$

$$(v_\eta)_u = -(v_\eta)_l$$

result in

$$w(x,y,z) = -K\iint_\tau \frac{1}{r}\left(B^2 u_\xi - v_\eta\right) d\xi\, d\eta \tag{3}$$

where τ represents one surface of the wing and of the wake that can influence the point x,y,z. The quantity v_η is a function of u through the irrotationality relation

$$v_\xi = u_\eta$$

Therefore,

$$v = \int_{\xi_L}^{x} u_\eta\, d\xi \tag{4a}$$

$$v_\eta = \int_{\xi_L}^{x} u_{\eta\eta}\, d\xi \tag{4b}$$

According to equation (3), the perturbation velocity w at any point in the flow field due to a lifting surface is determined by an integration of elementary source solutions with the distribution
2121 function given by $(B^2 u_\xi - v_\eta)$.

THE SEMI-INFINITE OBLIQUE WING WITH UNIFORM LOADING

An important application of equation (3) can be obtained by considering a semi-infinite oblique wing with uniform loading (fig. 1). The origin of the coordinate system is taken at the intersection of the leading and side edges, where both edges extend to infinity. For uniform loading, the term u_ξ in equation (3) vanishes everywhere over the region τ except across the leading edge. In evaluating the discontinuities in u or v that occur across the edges, a limiting procedure is used throughout the present analysis, which corresponds to the assumption of a linear variation in u or v across an infinitesimal strip of the edge. The distribution of v over the region τ for this type of loading, as obtained from equation (4a), is shown in figure 2, and v_η is seen to vanish everywhere except across the edges. In terms of equation (3), therefore, the downwash field for a semi-infinite oblique wing of uniform loading is obtained by means of a line integration of sources along the edges.

It is subsequently shown that the downwash solutions for the semi-infinite line sources along the leading and side edges may be used by simple manipulation to obtain the downwash field for finite plan forms of uniform loading.

The Semi-Infinite Oblique Leading Edge

The semi-infinite oblique leading edge with origin at (0,0,0) contributes to u_ξ and v_η along the edge and, by virtue of the discontinuity of the edge at the origin, contributes to v_η along the x-axis beginning at the origin and extending backwards to infinity. (See distribution of v in fig. 2.) The downwash contribution of the continuous portion of the leading edge is designated w_L; whereas the contribution of the edge discontinuity at the origin is designated w_{LD}.

Contribution of continuous portion of leading edge. - The integration of equation (3) along the oblique leading edge is most conveniently performed in terms of an oblique system of coordinates, such that 2121

$$\left.\begin{aligned} x' &= x-mB^2y \\ y' &= y-mx \\ z' &= z\sqrt{1-m^2B^2} \end{aligned}\right\} \tag{5}$$

It may be shown that the differential equation (2) is invariant under the change of variables $x \rightarrow x'$, $y \rightarrow y'$, and $z \rightarrow z'$ at both subsonic and supersonic speeds. If $\Omega(x,y,z)$ is a solution of equation (2), $\Omega(x',y',z')$ is therefore also a solution. For other examples of the use of the oblique transformation in wing-theory problems, see references 11 to 13.

In the oblique coordinate system, equation (3) evaluated along the wing leading edge becomes

$$w_L(x',y',z') = -K \int_L \frac{1}{r'} w_{z'} \Delta\eta' d\xi' \tag{6}$$

where

$$r' = \sqrt{(x'-\xi')^2 - B^2(y'^2+z'^2)}$$

Noting that

$$(w_{z'})_L = \frac{B^2u_{\xi'}-v_{\eta'}-mB^2(u_{\eta'}-v_{\xi'})}{\sqrt{1-m^2B^2}}$$

and along the leading edge for an infinitesimal strip of width $\Delta\eta'$,

$$u_{\xi'} = v_{\xi'} = 0$$

$$v_{\eta'}\Delta\eta' = \frac{u_0}{m}$$

$$u_{\eta'}\Delta\eta' = -u_0$$

results in

$$w_L(x',y',z') = K \frac{u_0\sqrt{1-B^2m^2}}{m} \int_0^{\xi_2'} \frac{d\xi'}{r'} \tag{7}$$

At subsonic speeds, the upper limit $\xi_2' = \infty$, and the integral is divergent. In actual cases, however, this divergence does not present any difficulty because the construction of the finite leading edge by means of the superposition of two semi-infinite leading edges of opposite sign leads to the result that the infinite upper limit cancels. At supersonic speeds, ξ_2' is the position of the last source with Mach aftercone including the point x',y',z'; that is,

$$\xi_2' = x'-B\sqrt{y'^2+z'^2}$$

Integration of equation (7) yields the following expressions for the line source originating at (0,0):

At subsonic speeds,

$$w_L(x,y,z) = \frac{u_0\sqrt{1+m^2B_1{}^2}}{2\pi m} \sinh^{-1} \frac{x'}{\xi_d'} \tag{8a}$$

where

$$\xi_d' = \sqrt{\left|B^2\right|(y'^2+z'^2)}$$

and

$$B_1 = \sqrt{1-M^2}$$

In equation (8a), the term that arises from substituting the upper limit of integration has been neglected because, as indicated previously, it vanishes for leading edges of finite lengths.

At supersonic speeds, for $Bm < 1$ (subsonic leading edge),

$$w_L(x,y,z) = \frac{u_0}{\pi m} \sqrt{1-B^2m^2} \cosh^{-1} \frac{x'}{\xi_d'} \tag{8b}$$

and for $Bm > 1$ (supersonic leading edge), equation (8b) becomes

$$w_L(x,y,z) = -\frac{u_0}{\pi m}\sqrt{B^2m^2-1}\,\cos^{-1}\frac{x'}{\xi_d'} \tag{8c}$$

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In equations (8) and in all subsequent expressions, the positive value in a radical term must be preserved when extracting the root. For example, if

$$y < 0$$

then

$$\sqrt{y^2} = \sqrt{(-y)^2} = -y = |y|$$

For a leading edge normal to the flight direction, $m = \infty$ and equations (8) yield the following expressions:

At subsonic speeds,

$$w_L(x,y,z) = \frac{B_1u_0}{2\pi}\sinh^{-1}\frac{B_1y}{\sqrt{x^2+B_1{}^2z^2}} \tag{8d}$$

At supersonic speeds ,

$$w_L(x,y,z) = -\frac{Bu_0}{\pi}\cos^{-1}\frac{-By}{\sqrt{x^2-B^2z^2}} \tag{8e}$$

It is shown by equations (8) that the downwash fields w_L contributed by the oblique and normal leading edges are conical, or w is constant along radial lines emanating from the origin.

Contribution of leading-edge discontinuity at origin. - From figure 2, it can be seen that the leading edge cut-off at the origin results in

$$v_\eta\Delta\eta = -\frac{u_0}{m} \tag{9a}$$

along the semi-infinite side edge. When equation (9a) is substituted into equation (3) (with u_ξ equal to zero along the side edge), there results

$$w_{LD}(x,y,z) = -\frac{Ku_0}{m}\int_0^{\xi_2}\frac{d\xi}{r} \tag{9b}$$

where ξ_2 is the position of the last source that can influence the point x,y,z; that is, at subsonic speeds, ξ_2 is at infinity; and at supersonic speeds,

$$\xi_2 = x-B\sqrt{y^2+z^2}$$

Integration of equation (9b) yields the following:

At subsonic speeds,

$$w_{LD}(x,y,z) = -\frac{u_0}{2\pi m}\sinh^{-1}\frac{x}{\xi_d} \tag{10a}$$

where the term arising from the infinite upper limit of integration has been neglected because it vanishes for leading edges of finite length and

$$\xi_d = \sqrt{|B^2|(y^2+z^2)}$$

At supersonic speeds,

$$w_{LD}(x,y,z) = -\frac{u_0}{\pi m}\cosh^{-1}\frac{x}{\xi_d} \tag{10b}$$

It is shown by equations (10) that the downwash contribution of the leading-edge discontinuity at the origin results entirely from the obliquity of the leading edge. Thus, if the leading edge is normal to the flight direction, this contribution vanishes.

The Semi-Infinite Streamwise Side Edge

Along the streamwise side edge, the quantity u_ξ vanishes; thus the source-distribution function along this edge is proportional only to the quantity v_η.

The details of evaluating the contribution of the streamwise side edge to the downwash field are given in the appendix. The results for the semi-infinite left streamwise side edge are as follows:

At subsonic speeds,

$$w_S(x,y,z) = -\frac{B_1^{\,2}u_0 y}{2\pi}\left(\frac{1}{x+\sqrt{x^2+\xi_d^{\,2}}} + \frac{2x}{\xi_d^{\,2}}\right) \tag{11a}$$

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At supersonic speeds,

$$w_S(x,y,z) = \frac{B^2 u_0 y}{\pi}\left(\frac{1}{x+\sqrt{x^2-\xi_d^{\,2}}} - \frac{x}{\xi_d^{\,2}}\right) \tag{11b}$$

Equations (11) show that the downwash contribution of the streamwise side edge is independent of the obliquity of the leading edge.

FINITE WINGS WITH UNIFORM LOADING

The downwash field due to finite wings with uniform loading can be obtained by superimposing the fields due to a number of semi-infinite wings of the type considered in the previous section (fig. 1). This superposition is equivalent to superimposing the fields due to the source lines expressed by equations (8), (10), and (11).

A plan form with curved edges requires an infinite number of source lines. If the edges are composed of straight-line segments, however, a finite number of source lines can be used to represent the plan form.

Swept Wings with Streamwise Tips

Let the downwash field due to a semi-infinite line source beginning at x_0,y_0 be denoted by w_{x_0,y_0}; let the subscripts L, S, T, LD, and TD refer to the leading, side, and trailing edges and the leading- and trailing-edge discontinuities, respectively, all for the right half-wing; and let $\overline{w}$ refer to the effect of the source lines originating on the left half-wing. Then, the downwash field for a uniformly loaded wing, which is symmetrical with respect to the x-axis and which has uniformly swept leading edges and streamwise tips, is given by the following sum:

$$w_0 = \left[(w+\overline{w})_{0,0} - (w+\overline{w})_{\frac{h}{m},\pm h}\right]_{L,LD} - \left[(w+\overline{w})_{c_r,0} - (w+\overline{w})_{\frac{h}{m}+c_t,\pm h}\right]_{T,TD} -$$

$$\left[(w+\overline{w})_{\frac{h}{m},\pm h} - (w+\overline{w})_{\frac{h}{m}+c_t,\pm h}\right]_S \tag{12a}$$

In equation (12a) and in the subsequent expressions, the upper and lower signs preceding a term refer to the right and left half-wings, respectively.

The downwash fields due to the semi-infinite line sources indicated in equation (12a) can be obtained by simple manipulation of equations (8), (10), and (11). The following transformations are made to obtain the effect of a semi-infinite line source originating at x_0, y_0:

$$\left.\begin{aligned} x \text{ in equations (8), (10), and (11)} &= x - x_0 \\ y \text{ in equations (8), (10), and (11)} &= \pm(y - y_0) \end{aligned}\right\} \tag{12b}$$

Equations (8) and (10) can also be applied to a semi-infinite line source along the trailing edge by replacing m in these equations with the cotangent of the sweep angle of the trailing edge.

At subsonic speeds, every point in the field is affected by all the terms in equation (12a); whereas at supersonic speeds, the point is affected only by those terms that refer to edges which lie within the Mach forecone from the point under consideration.

Rectangular Wings

For the rectangular wing, the edge discontinuities LD and TD disappear and equation (12a) becomes

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$$w_0 = \Big[(w+\overline{w})_{0,0} - (w+\overline{w})_{0,\pm h}\Big]_L - \Big[(w+\overline{w})_{c,0} - (w+\overline{w})_{c,\pm h}\Big]_T -$$

$$\Big[(w+\overline{w})_{0,\pm h} - (w+\overline{w})_{c,\pm h}\Big]_S \tag{12c}$$

WINGS WITH ARBITRARY LOADING

If the wing loading is an arbitrary function of x and y, the integration in equation (3) for the downwash field is, in general, required over the entire wing surface and in the wake. An alternative procedure in this case is the use of "correspondence formulas" as indicated in reference 8 by means of which the downwash field due to a variable loading may be expressed in terms of the downwash field due to a uniform loading with the addition of corrections for the edges of the plan form.

In reference 8, it is shown that if u on the wing is expressed in a series as a function of x and y in terms of the uniform prescribed velocity u_0 and any term in the series is differentiated with respect to x and y, s and t times, respectively, such that

$$(u)_{x^s y^t} = bu_0 \text{ (on wing)} \tag{13a}$$

where b is a constant; then subject to edge corrections, there results

$$(w_{n,k})_{x^s y^t}(x,y,z) = bw_0(x,y,z) \tag{13b}$$

where $w_{n,k}$ and w_0 refer to the downwash fields due to the term in the series and to a uniform loading, respectively. The edges of finite plan forms may alter the given relation (13a) so that edge corrections may be required for the relation expressed by equation (13b). At supersonic speeds, the relation given by equation (13b) thus applies to finite wings at all points in the flow field outside of the Mach aftercone from the edges that alter the given relation (13a) on the wing. Correspondence formulas for rectangular wings at subsonic and supersonic speeds are given in reference 8, table I.

ILLUSTRATIVE EXAMPLE FOR RECTANGULAR WING

AT SUPERSONIC SPEEDS

As an illustration of the method described herein, the downwash field is obtained at supersonic speeds for a rectangular wing with uniform loading and with linear chordwise variation in loading. The lettered regions in the subsequent discussion refer to figure 3.

Uniform Loading

For the rectangular wing with uniform loading at supersonic speed, equations (8e), (11b), and (12c) are applicable.

Region A. - In the region within the leading-edge Mach cones and outside of the side-edge and trailing-edge Mach cones,

$$(w_0)_A(x,y,z) = (w_{0,0}+\bar{w}_{0,0})_L$$

$$= -\frac{Bu_0}{\pi}\left(\cos^{-1}\frac{-By}{\sqrt{x^2-B^2z^2}} + \cos^{-1}\frac{By}{\sqrt{x^2-B^2z^2}}\right) = -Bu_0 \qquad (14)$$

Region B. - In the region within the trailing-edge Mach cones and outside of the side-edge Mach cones,

$$(w_0)_B(x,y,z) = (W_0)_A - (w_{c,0} + \bar{w}_{c,0})_T$$

$$= -Bu_0 + \frac{Bu_0}{\pi}\left(\cos^{-1}\frac{-By}{\sqrt{(x-c)^2-B^2z^2}} + \cos^{-1}\frac{By}{\sqrt{(x-c)^2-B^2z^2}}\right) = 0 \qquad (15)$$

Region C. - Region C, which refers to points outside the trailing-edge Mach cones, is divided into two subregions depending on whether the point lies within one or both side-edge Mach cones.

Region C_1. - If the point lies within one side-edge Mach cone,

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$$
\begin{aligned}
(w_0)_{C_1}(x,y,z) &= (w_0)_A - (w_{0,h})_L - (w_{0,h})_S \\
&= (w_0)_A + \frac{Bu_0}{\pi}\left[\cos^{-1}\frac{-By_a}{\sqrt{x^2-B^2z^2}} - By_a\left(\frac{1}{x+\sqrt{x^2-\xi_a{}^2}} - \frac{x}{\xi_a{}^2}\right)\right] \\
&= -\frac{Bu_0}{\pi}\left[\cos^{-1}\frac{By_a}{\sqrt{x^2-B^2z^2}} + By_a\left(\frac{1}{x+\sqrt{x^2-\xi_a{}^2}} - \frac{x}{\xi_a{}^2}\right)\right]
\end{aligned}
\tag{16}
$$

where

$$\xi_a = B\sqrt{y_a{}^2+z^2}$$

and

$$y_a = y-h$$

Region C_2. - For points within both side-edge Mach cones,

$$
\begin{aligned}
(w_0)_{C_2}(x,y,z) &= (w_0)_{C_1} - (\bar{w}_{0,-h})_L - (\bar{w}_{0,-h})_S \\
&= (w_0)_{C_1} + \frac{Bu_0}{\pi}\left[\cos^{-1}\frac{By_b}{\sqrt{x^2-B^2z^2}} + By_b\left(\frac{1}{x+\sqrt{x^2-\xi_b{}^2}} - \frac{x}{\xi_b{}^2}\right)\right]
\end{aligned}
\tag{17}
$$

where

$$\xi_b = B\sqrt{y_b{}^2+z^2}$$

and

$$y_b = y+h$$

Region D. - Region D refers to points that are always within Mach cones from the leading, side, and trailing edges. This region is subdivided into five regions, as shown in figure 3.

Region D_1. - For points in D_1,

$$(w_0)_{D_1}(x,y,z) = (w_0)_{C_1} - (w_{c,0} + \bar{w}_{c,0})_T = (w_0)_{C_1} + Bu_0 \tag{18}$$

Region D_2. - For points in D_2,

$$(w_0)_{D_2}(x,y,z) = (w_0)_{C_1} + Bu_0 + (w_{c,h})_T + (w_{c,h})_S$$

$$= (w_0)_{C_1} + \frac{Bu_0}{\pi}\left[\cos^{-1}\frac{By_a}{\sqrt{(x-c)^2-B^2z^2}} + By_a\left(\frac{1}{x-c+\sqrt{(x-c)^2-\xi_a{}^2}} - \frac{x-c}{\xi_a{}^2}\right)\right] \tag{19}$$

Region D_3. - For points in D_3,

$$(w_0)_{D_3}(x,y,z) = (w_0)_{D_1} - (\bar{w}_{0,-h})_L - (\bar{w}_{0,-h})_S$$

$$= (w_0)_{D_1} + \frac{Bu_0}{\pi}\left[\cos^{-1}\frac{By_b}{\sqrt{x^2-B^2z^2}} + By_b\left(\frac{1}{x+\sqrt{x^2-\xi_b{}^2}} - \frac{x}{\xi_b{}^2}\right)\right] \tag{20}$$

Region D_4. - For points in D_4,

$$(w_0)_{D_4}(x,y,z) = (w_0)_{D_2} - (\bar{w}_{0,-h})_L - (\bar{w}_{0,-h})_S = (w_0)_{D_2} + (w_0)_{D_3} - (w_0)_{D_1} \tag{21}$$

Region D_5. - For points in D_5,

$$(w_0)_{D_5}(x,y,z) = (w_0)_{D_4} + (\bar{w}_{c,-h})_T + (\bar{w}_{c,-h})_S$$

$$= (w_0)_{D_4} - \frac{Bu_0}{\pi}\left[\cos^{-1}\frac{By_b}{\sqrt{(x-c)^2-B^2z^2}} + By_b\left(\frac{1}{x-c+\sqrt{(x-c)^2-\xi_b{}^2}} - \frac{x-c}{\xi_b{}^2}\right)\right] \tag{22}$$

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Linear Chordwise Loading

For linear chordwise loading, let

$$u_1 = au_0 x \quad \text{(on wing)} \tag{23a}$$

where a and u_0 are constants. The downwash field for this type of loading can be obtained by using the correspondence formulas of reference 8 in conjunction with the downwash field obtained in the preceding section for the uniformly loaded rectangular wing at supersonic speed. If w_1 and w_0 refer to the downwash fields due to linear chordwise loading and to uniform loading, respectively, then from reference 8, table I, for all points outside the trailing-edge Mach cones,

$$w_1(x,y,z) = a\int_{Bz}^{x} w_0(\xi,y,z)\, d\xi \tag{23b}$$

and for points within the trailing-edge Mach cones,

$$w_1(x,y,z) = a\left[\int_{Bz}^{x} w_0(\xi,y,z)\, d\xi + c(\Delta w_0)_T(x,y,z)\right] \tag{23c}$$

where $(\Delta w_0)_T$ refers to the effect of the wing cut-off at the trailing edge for uniform loading.

Region A. - For the region within the leading-edge Mach cone and outside the side-edge and trailing-edge Mach cones, equations (23b) and (14) are applicable and there results

$$(w_1)_A(x,y,z) = -aBu_0\,(x-Bz) \tag{24}$$

Region B. - In the region within the trailing-edge Mach cones and outside the side-edge Mach cones, equations (23c), (14), and (15) yield

$$(w_1)_B(x,y,z) = -aBu_0\int_{Bz}^{c+Bz} d\xi + acBu_0 = 0 \tag{25}$$

Region C. - In the region within one or both side-edge Mach cones and outside the trailing-edge Mach cones, equation (23b) applies.

Region C_1. - For points in region C_1, equation (23b) may be written

$$(w_1)_{C_1}(x,y,z) = a\int_{Bz}^{x} w_0(\xi,y,z)\,d\xi = a\int_{Bz}^{x}(w_0)_{C_1}(\xi,y,z)\,d\xi$$

$$= -\frac{aBu_0}{\pi}\left[\int_{Bz}^{x}\left(\cos^{-1}\frac{By_a}{\sqrt{\xi^2-B^2z^2}}\right)d\xi + By_a\int_{\xi_a}^{x}\left(\frac{1}{\xi+\sqrt{\xi^2-\xi_a{}^2}} - \frac{\xi}{\xi_a{}^2}\right)d\xi\right] \tag{26a}$$

In equation (26a), $w_0(\xi,y,z)$ is represented as $(w_0)_{C_1}$ (equation (16)) throughout the entire range of integration because $(w_0)_{C_1}$ evaluated in region A is equal to $(w_0)_A$ (equation (14)), inasmuch as the imaginary part is discarded.

Equation (26a) requires the evaluation of the following integrals:

$$I_1 = \int_{Bz}^{x}\cos^{-1}\frac{By_a}{\sqrt{\xi^2-B^2z^2}}\,d\xi \tag{26b}$$

where By_a and B^2z^2 are constants, and

$$I_2 = \int_{\xi_a}^{x}\left(\frac{1}{\xi+\sqrt{\xi^2-\xi_a{}^2}} - \frac{\xi}{\xi_a{}^2}\right)d\xi \tag{26c}$$

where ξ_a is a constant.

The solution for I_1 obtained by integration by parts and simplification is

$$I_1(x,y_a,z) = x \cos^{-1} \frac{By_a}{\sqrt{x^2-B^2z^2}} - By_a \cosh^{-1} \frac{x}{\xi_a} - Bz \cos^{-1} \frac{By_a x}{\xi_a \sqrt{x^2-B^2z^2}} \tag{26d}$$

The solution for I_2 is

$$I_2(x,y_a,z) = \frac{1}{2}\left[\cosh^{-1} \frac{x}{\xi_a} - \frac{x\sqrt{x^2-\xi_a{}^2}}{\xi_a{}^2}\right] \tag{26e}$$

Utilizing the solutions for I_1 and I_2 given by equations (26d) and (26e), equation (26a) yields for points in region C within one side-edge Mach cone,

$$(w_1)_{C_1}(x,y,z) = -\frac{aBu_0}{\pi}\left[I_1(x,y_a,z) + By_a I_2(x,y_a,z)\right] \tag{26f}$$

Region C_2. - For points in region C within both side-edge Mach cones, equation (23b) may be written

$$(w_1)_{C_2}(x,y,z) = a\int_{Bz}^{x} w_0(\xi,y,z)\, d\xi$$

$$= (w_1)_{C_1} + \frac{aBu_0}{\pi}\left[\int_{Bz}^{x} \cos^{-1} \frac{By_b}{\sqrt{\xi^2-B^2z^2}}\, d\xi + By_b \int_{\xi_b}^{x}\left(\frac{1}{\xi+\sqrt{\xi^2-\xi_b{}^2}} - \frac{\xi}{\xi_b{}^2}\right) d\xi\right] \tag{27a}$$

where $w_0(\xi,y,z)$ has been obtained from equations (16) and (17). Utilizing the solutions for I_1 and I_2 given in equations (26d) and (26e), in which y_a and ξ_a are replaced by y_b and ξ_b, respectively, changes equation (27a) to yield

$$(w_1)_{C_2}(x,y,z) = (w_1)_{C_1} + \frac{aBu_0}{\pi}\left[I_1(x,y_b,z) + By_b\, I_2(x,y_b,z)\right] \tag{27b}$$

Region D. - In the region within the trailing-edge Mach cone and within one or both side-edge Mach cones, equation (23c) applies.

Region D_1. - For points in region D_1, the integral in equation (23c) becomes

$$a\int_{Bz}^{x} w_0(\xi,y,z)\,d\xi = (w_1)_{C_1} + aBu_0\int_{c+Bz}^{x} d\xi = (w_1)_{C_1} + aBu_0\,(x-c-Bz) \tag{28a}$$

where $w_0(\xi,y,z)$ has been obtained from equations (16) and (18).

The quantity $(\Delta w_0)_T$ in equation (23c) is evaluated for this case as the difference between $(w_0)_{D_1}$ and $(w_0)_{C_1}$; that is, from equation (18), $(\Delta w_0)_T = Bu_0$. When these results are combined, equation (23c) yields

$$(w_1)_{D_1}(x,y,z) = (w_1)_{C_1} + aBu_0\,(x-Bz) \tag{28b}$$

Region D_2. - For points in region D_2, the integral in equation (23c) becomes

$$a\int_{Bz}^{x} w_0(\xi,y,z)\,d\xi = (w_1)_{C_1} + \frac{aBu_0}{\pi}\left[\int_{c+Bz}^{x} \cos^{-1}\frac{By_a}{\sqrt{(\xi-c)^2-B^2z^2}}\,d\xi\ + \right.$$

$$\left. By_a\int_{c+\xi_a}^{a}\left(\frac{1}{\xi-c+\sqrt{(\xi-c)^2-\xi_a{}^2}} - \frac{\xi-c}{\xi_a{}^2}\right)d\xi\right] \tag{29a}$$

where $w_0(\xi,y,z)$ has been obtained from equations (16) and (19). The integrals in equation (29a) may be evaluated by utilizing the solutions for I_1 and I_2 as expressed by equations (26d) and (26e). In these solutions x is replaced by $x-c$. The quantity $(\Delta w_0)_T$ in equation (23c) is evaluated in this case as the difference between $(w_0)_{D_2}$ and $(w_0)_{C_1}$. When these results are combined, equation (23c) yields

$$(w_1)_{D_2}(x,y,z) = (w_1)_{C_1} + \frac{aBu_0}{\pi}\Bigg\{ I_1(x-c,y_a,z) + By_a I_2(x-c,y_a,z) + c\left[\cos^{-1}\frac{By_a}{\sqrt{(x-c)^2-B^2z^2}} + By_a\left(\frac{1}{x-c+\sqrt{(x-c)^2-\xi_a^2}} - \frac{x-c}{\xi_a^2}\right)\right]\Bigg\} \tag{29b}$$

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Region D_3. - For points in region D_3, equation (23c) yields

$$(w_1)_{D_3}(x,y,z) = (w_1)_{D_1} + \frac{aBu_0}{\pi}\left[\int_{Bz}^{x} \cos^{-1}\frac{By_b}{\sqrt{\xi^2-B^2z^2}}\,d\xi + By_b\int_{\xi_b}^{x}\left(\frac{1}{\xi+\sqrt{\xi^2-\xi_b^2}} - \frac{\xi}{\xi_b^2}\right)d\xi\right] \tag{30a}$$

where $w_0(\xi,y,z)$ has been obtained from equations (18) and (20) and the term $(\Delta w_0)_T$ is included in $(w_1)_{D_1}$. The integrals in equation (30a) may be evaluated by utilizing the solutions for I_1 and I_2 as expressed by equations (26d) and (26e). In these solutions, y_a is replaced by y_b; thus

$$(w_1)_{D_3}(x,y,z) = (w_1)_{D_1} + \frac{aBu_0}{\pi}\left[I_1(x,y_b,z) + By_b\, I_2(x,y_b,z)\right] \tag{30b}$$

Region D_4. - For points in region D_4, equation (23c) yields

$$(w_1)_{D_4}(x,y,z) = (w_1)_{D_2} + (w_1)_{D_3} - (w_1)_{D_1} \tag{31}$$

where $w_0(\xi,y,z)$ has been obtained from equations (19) and (21).

Region D_5. - For points in region D_5, equation (23c) yields

$$(w_1)_{D_5}(x,y,z) = (w_1)_{D_4} - \frac{aBu_0}{\pi}\left\{\int_{c+Bz}^{x} \cos^{-1} \frac{By_b}{\sqrt{(\xi-c)^2-B^2z^2}}\, d\xi + \right.$$

$$By_b \int_{c+\xi_b}^{x} \left(\frac{1}{\xi-c+\sqrt{(\xi-c)^2-\xi_b{}^2}} - \frac{\xi-c}{\xi_b{}^2}\right) d\xi +$$

$$\left. c\left[\cos^{-1} \frac{By_b}{\sqrt{(x-c)^2-B^2z^2}} + By_b\left(\frac{1}{x-c+\sqrt{(x-c)^2-\xi_b{}^2}} - \frac{x-c}{\xi_b{}^2}\right)\right]\right\} \tag{32a}$$

where $w_0(\xi,y,z)$ has been obtained from equation (22). The integrals in equation (32a) may be evaluated by utilizing the solutions for I_1 and I_2 as expressed by equations (26d) and (26e). In these solutions, x is replaced by $x-c$, and y_a and ξ_a are replaced by y_b and ξ_b, respectively. Thus

$$(w_1)_{D_5}(x,y,z) = (w_1)_{D_4} - \frac{aBu_0}{\pi}\left\{ I_1(x-c,y_b,z) + By_b\, I_2(x-c,y_b,z) + \right.$$

$$\left. c\left[\cos^{-1} \frac{By_b}{\sqrt{(x-c)^2-B^2z^2}} + By_b\left(\frac{1}{x-c+\sqrt{(x-c)^2-\xi_b{}^2}} - \frac{x-c}{\xi_b{}^2}\right)\right]\right\} \tag{32b}$$

Lewis Flight Propulsion Laboratory,
National Advisory Committee for Aeronautics,
Cleveland, Ohio, January 22, 1951.

APPENDIX

DERIVATION OF DOWNWASH FIELD DUE TO SEMI-INFINITE LEFT STREAMWISE SIDE EDGE

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The subsequent derivation for the downwash field refers to the left streamwise side edge of the uniformly loaded semi-infinite oblique wing, as shown in figure 1.

The downwash field contributed by the side edge may be obtained by means of equation (3), in which the integral is evaluated along the side edge. As noted in the text, the quantity u_ξ is zero along the streamwise side edge (of width $\Delta\eta$); therefore,

$$w_S(x,y,z) = K\int_S \frac{v_\eta \Delta\eta \, d\xi}{r} \tag{A1}$$

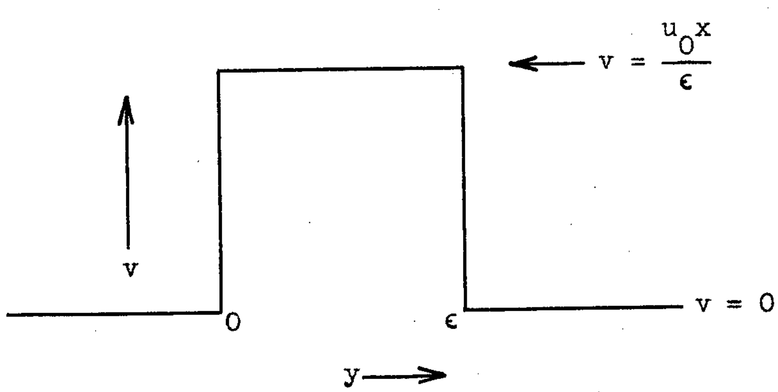
Across the side edge there is an abrupt increase in u. This increase is assumed to occur over an infinitesimal width ϵ across the side edge. Then according to equation (4a),

$$v(x,0,0) = \int_0^x u_\eta \, d\xi = \frac{u_0}{\epsilon} x \tag{A2}$$

At $y = \epsilon$, however, u_η due to the side edge vanishes and therefore

$$v(x,\epsilon,0) = 0 \tag{A3}$$

The distribution of v in the vicinity of the side edge $(y = 0)$ is shown in the following sketch:



It is assumed that the abrupt changes in v indicated in the sketch occur over the infinitesimal distances $\Delta\eta$, so that along the line $(\xi,0,0)$,

$$v_\eta \Delta\eta = \frac{u_0 \xi}{\epsilon} \tag{A4}$$

and along the line $(\xi,\epsilon,0)$,

$$v_\eta \Delta\eta = -\frac{u_0 \xi}{\epsilon} \tag{A5}$$

Substituting these values for $v_\eta \Delta\eta$ into equation (A1) results in

$$w_S(x,y,z) = \frac{Ku_0}{\epsilon}\left[\int_0^{\infty,\, x-B\sqrt{y^2+z^2}} \frac{\xi\, d\xi}{\sqrt{(x-\xi)^2-B^2(y^2+z^2)}} - \int_0^{\infty,\, x-B\sqrt{(y-\epsilon)^2+z^2}} \frac{\xi\, d\xi}{\sqrt{(x-\xi)^2-B^2(y-\epsilon)^2+z^2}}\right] \tag{A6}$$

where the two upper limits refer to subsonic and supersonic speeds, respectively. Integration of equation (A6) yields

$$w_S(x,y,z) = \frac{Ku_0}{\epsilon}\Bigg(\left\{-x \log\left|x-\xi + \sqrt{(x-\xi)^2-B^2(y^2+z^2)}\right| + \sqrt{(x-\xi)^2-B^2(y^2+z^2)}\right\}_0^{\infty,\, x-B\sqrt{y^2+z^2}} + \left\{x \log\left|x-\xi + \sqrt{(x-\xi)^2-B^2\left[(y-\epsilon)^2+z^2\right]}\right| - \sqrt{(x-\xi)^2-B^2\left[(y-\epsilon)^2+z^2\right]}\right\}_0^{\infty,\, x-B\sqrt{(y-\epsilon)^2+z^2}}\Bigg) \tag{A7}$$

When the limits are substituted into equation (A7) and ϵ is made to approach zero, the following results are obtained:

At subsonic speeds,

$$w_S(x,y,z) = \frac{B_1{}^2 u_0 y}{2\pi}\left(-\frac{1}{x+\sqrt{x^2+\xi_d{}^2}} - \frac{2x}{\xi_d{}^2}\right) \tag{A8}$$

where

$$\xi_d = \sqrt{|B^2|(y^2+z^2)}$$

At supersonic speeds,

$$w_S(x,y,z) = \frac{B^2 u_0 y}{\pi}\left(\frac{1}{x+\sqrt{x^2-\xi_d{}^2}} - \frac{x}{\xi_d{}^2}\right) \tag{A9}$$

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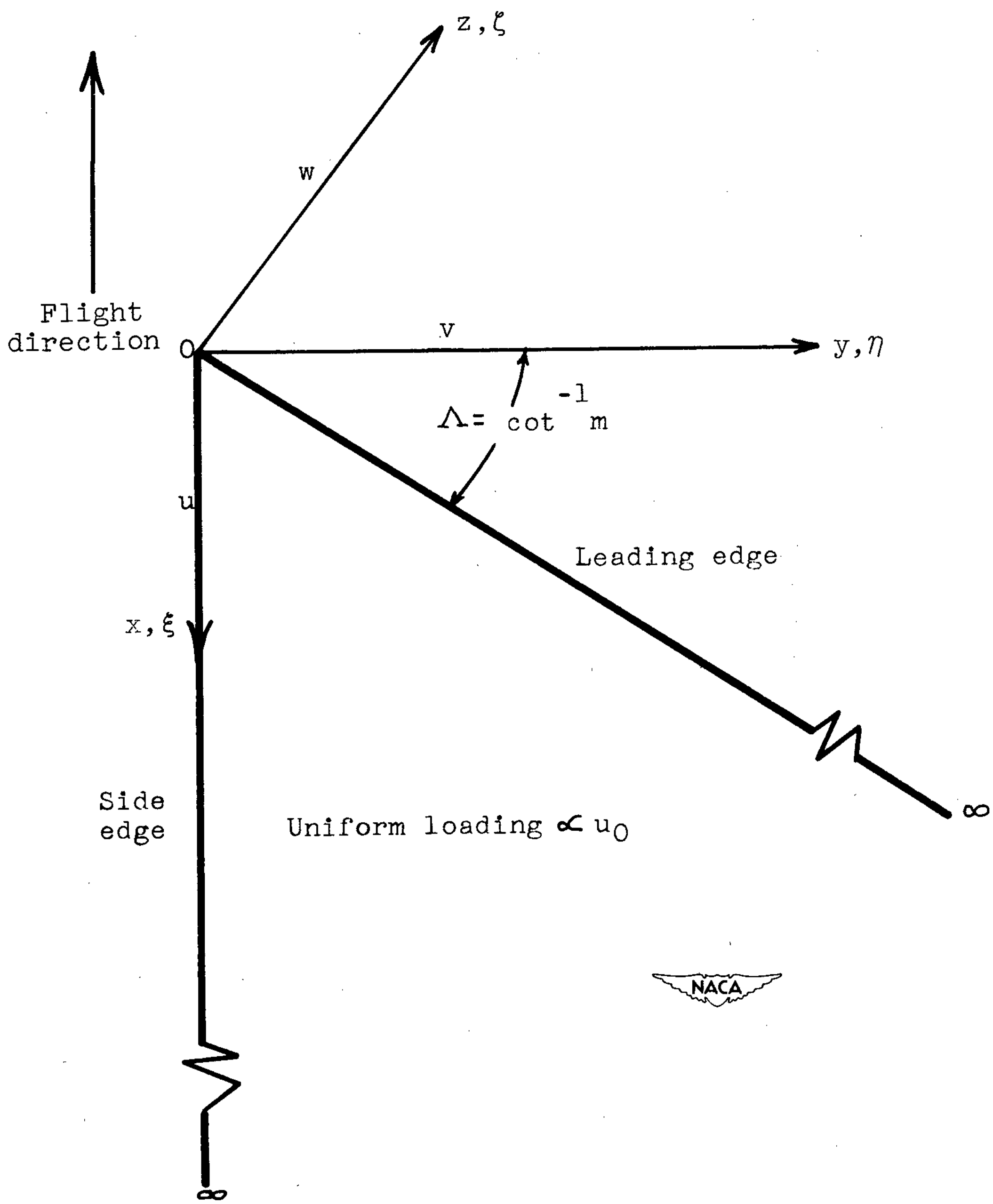

Figure 1. - Semi-infinite oblique wing with uniform loading. Origin is at leading edge of center section.

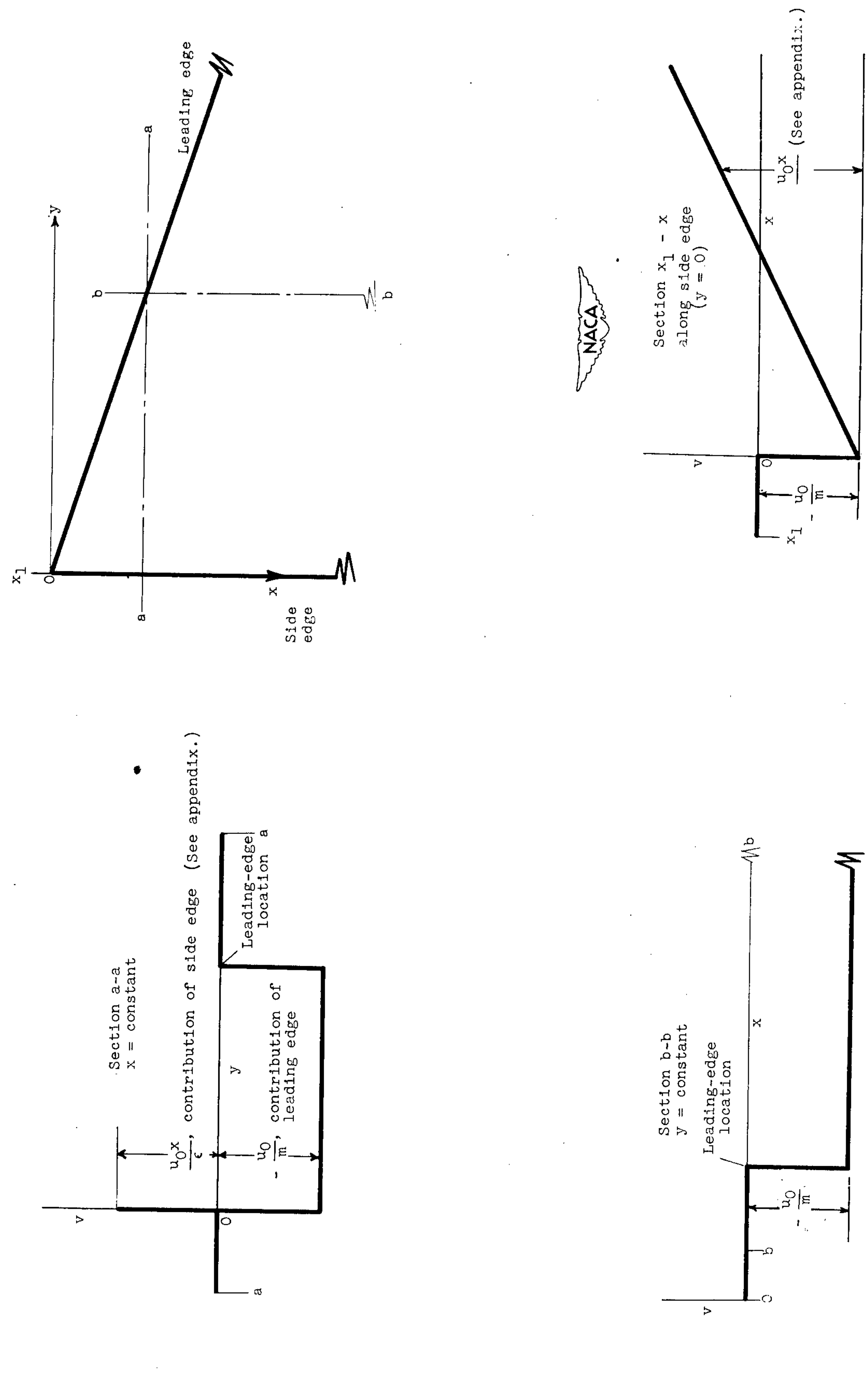

Figure 2. - Distribution of v on upper surface of semi-infinite oblique wing with uniform loading.

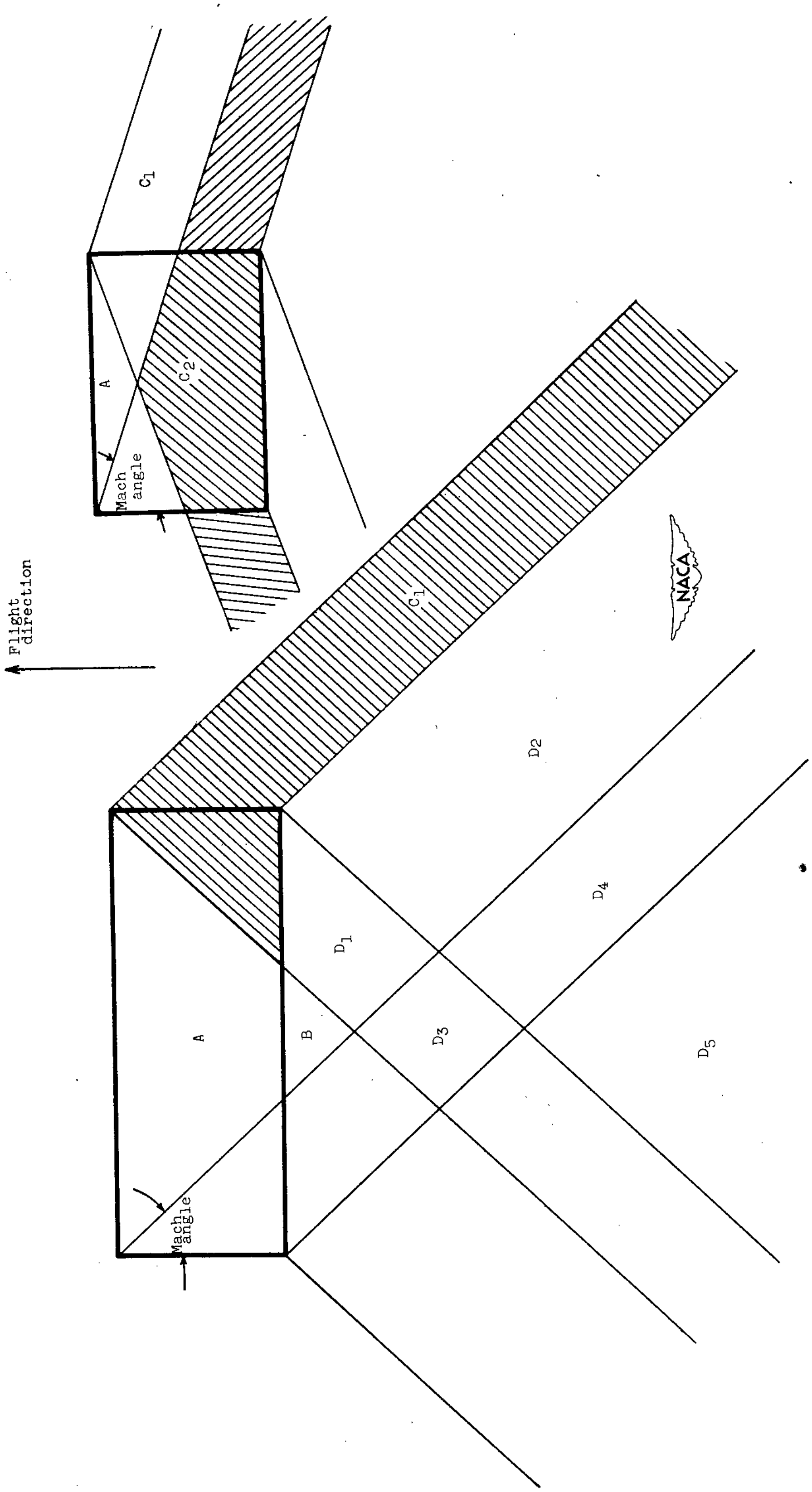

Figure 3. - Plan view of regions considered in examples for rectangular wing at supersonic speeds. Regions bounded by portions of three-dimensional Mach aft cones.

Abstract

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Flow, Subsonic 1.1.2.1 S

Method for Calculating Downwash Field due to Lifting Surfaces at Subsonic and Supersonic Speeds.

By Sidney M. Harmon.

NACA TN 2344

April 1951

(Abstract on Reverse Side)

Flow, Supersonic 1.1.2.3 S

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NACA TN 2344

April 1951

(Abstract on Reverse Side)

Wings, Complete - Theory 1.2.2.1

Method for Calculating Downwash Field due to Lifting Surfaces at Subsonic and Supersonic Speeds.

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April 1951

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Wings, Complete - Sweep 1.2.2.2.3

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